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Optical potentials in algebraic scattering theory

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Abstract. Using the theory of induced representations new realizations for the Lie algebras of the groups $SO(2,1)$, $SO(2,2)$, $SO(3,2)$ are found. The eigenvalue problem of the Casimir operators yield Schrödinger equations with non-Hermitian interaction terms (i.e. optical potentials). For the group $SO(2,2)$ we have a two-parameter family of (matrix-valued) potentials containing terms of Pöschl–Teller and Gendenshtein type. We calculate the *S*-matrices for special values of this two-parameter family. In particular we also include a derivation of the *S*-matrix for the *two-dimensional* scattering problem on a *complex* Gendenshtein potential. The canonically transformed realization results in a non-local optical potential.

1. Introduction

During the past two decades spectrum generating algebras and dynamic symmetries proved to be valuable theoretical tools for the study of bound-state and scattering problems. The earliest applications of this kind included the study of collective states in nuclei, and the rotations and vibrations of molecules. More recently, with the advent of algebraic scattering theory (AST), dynamic symmetries of scattering problems have been extensively used to describe heavy ion reactions. (For a brief review on dynamic symmetries and their applications to scattering problems see the paper by Iachello in [1] and references therein.) The main theme in these investigations is the determination of the interaction term governing the scattering process compatible with the algebraic *S*-matrix fixed by the mathematical structure of the *non-compact* dynamic symmetry group G. In particular the *S*-matrix corresponding to the group $SO(3,2)$ turned out to be suitable for the description of the observed experimental data in heavy ion reactions [2]. In order to know more about the origin of such symmetries for these systems, it is important to find new coordinate realizations of the possible symmetry groups G. The desirable realizations are those which can account for the basic properties of the interaction known from experiments. These properties can be described by interaction terms which are of the form of an energy- and spin-dependent multichannel, non-local optical potential of modified Coulomb type.

The purpose of the present paper is to report on the existence of non-standard realizations found for the Lie algebras of the groups $SO(2,1)$, $SO(2,2)$, $SO(3,2)$ capable of achieving this goal. The possibility of using such techniques was announced in [3], and further developed in a series of papers [4–7]. In [3] we explained the meaning of non-standard realizations using the theory of induced representations. The basic idea was to choose a finite-dimensional *matrix representation* of a *compact* subgroup H of G, and then use the inducing construction to arrive at a realization of **g** (the Lie algebra of G) in terms of *matrix-valued differential operators*. Employing the groups $SO(n,1)$ in [5, 6] we have shown that using this method scattering problems including spin can naturally be described within the framework of AST. A further step was made in [7], where we also managed to derive *non-local* potentials with modified Coulomb behaviour for the group $SO(3,1)$. In these papers the inducing subgroup was *compact* ($SO(n)$). In a recent paper (see our paper in [1]) we reported on the existence of a realization for the group $SO(3,2)$ we accidentally came across, yielding *non-Hermitian interaction terms* (i.e. optical potentials). Such potentials, according to the optical model [8], describe energy dissipation, hence they are used extensively, for example, in models of inelastic processes of nuclear reactions. The important observation of this paper was, that such terms appeared because in this case the inducing subgroup was *non-compact*. We conjectured that generally, the presence of *non-Hermitian interaction terms* might be traced back to the choice of a *finite-dimensional* (hence *non-unitary*) inducing representation of the non-compact subgroup. In this paper, by considering the groups $SO(2,1)$, $SO(2,2)$, $SO(3,2)$ we show that the presence of optical potentials can really be traced back to the choice of finite-dimensional non-unitary representations for the subgroups $SO(1,1)$, $SO(2,1)$, and $SO(3,1)$ of the aforementioned groups, and then inducing a representation for them.

We emphasize here that these representations and those found in this paper are known in the mathematics literature. The explicit form of their generators, however, has not been constructed. In [3] we showed that there exists an explicit construction for these matrix-valued generators using the geometric properties of coset spaces G/H. Here, however, we have found it more instructive to follow a different route, namely guessing the right form of modification that has to be added to the realizations widely used in AST and then checking the commutation relations. A guiding line for this pedestrian method is provided by the simplest $SO(2,1)/SO(1,1)$ case where the explicit method of [3] is easily applied, and the basic patterns are clearly recognized. For the more complicated cases the equivalence with an explicit construction will be justified in a subsequent publication.

The organization of this paper is as follows. In section 2 we investigate the simplest non-compact group with a non-compact subgroup, namely $SO(2,1)$. In section 3 we generalize our results for $SO(2,2)$ with the non-Abelian non-compact subgroup $SO(2,1)$. In these sections we clarify the general structure of the matrix-valued realizations and their connection to non-Hermitian interaction terms. The (matrix-valued) potentials contain terms of Pöschl–Teller and Gendenshtein type. In section 4 we calculate the S-matrices for special cases giving real or complex interaction terms. In section 5 a straightforward generalization of our construction for the group $SO(3,2)$ favoured by heavy ion physicists is given. Conclusions and some comments are given in section 6. Here we also give some hints for obtaining non-local optical potentials by the method of canonical transformations. In appendix A we discuss some geometric properties of our modified generators using the language of the theory of induced representations. As far as we know the calculation of the S-matrix for the *two-dimensional* scattering problem on a *complex* Gendenshtein potential has not been given. We include this derivation in appendix B.

2. A realization for $SO(2,1)$ using coordinates on the coset $SO(2,1)/SO(1,1)$

In order to motivate our construction, we start with the simplest example of a representation for a non-compact group G induced by a representation $\mathcal{D}$ of a *non-compact* subgroup H of G. Hence we choose $G = SO(2,1)$ and $H = SO(1,1)$. According to [3] the generators of the Lie algebra $so(2,1)$ we are searching for are matrix-valued differential operators, expressed in terms of suitable coordinates on the coset space $SO(2,1)/SO(1,1)$. Moreover, the matrix-valued nature of the generators is fixed by a representation for the subgroup $SO(1,1)$. Since this group is Abelian, we have a 1×1 matrix, i.e. a number for this representation. Let us denote this number by q, i.e. we have $q \in \mathbb{R}$ when the representation is unitary, but we also allow the possibility for allowing $q \in \mathbb{C}$ [9] for non-unitary inducing representations.

2.1. The case with trivial inducing representation

First we consider the case $q = 0$. In this case we have the usual differential operators corresponding to the infinitesimal action of $SO(2,1)$ on our coset which is isomorphic to the *one-sheet hyperboloid* [9]. This is described by the constraint ${X_1}^2 + {X_2}^2 - {X_3}^2 = 1$. Our differential operators are then

$$L_1 = -\mathrm{i}\left(X_2\frac{\partial}{\partial X_3} + X_3\frac{\partial}{\partial X_2}\right) \qquad L_2 = \mathrm{i}\left(X_3\frac{\partial}{\partial X_1} + X_1\frac{\partial}{\partial X_3}\right) \tag{2.1a}$$

$$L_3 = -\mathrm{i}\left(X_1\frac{\partial}{\partial X_2} - X_2\frac{\partial}{\partial X_1}\right). \tag{2.1b}$$

A convenient coordinatization of our hyperboloid is given by

$$X_1 = \frac{1+R^2}{1-R^2}v_1 \qquad X_2 = \frac{1+R^2}{1-R^2}v_2 \qquad X_3 = \frac{2R}{1-R^2} \tag{2.2}$$

with

$$v_1 = \cos\chi \qquad v_2 = \cos\chi \qquad v_3 = -\mathrm{i}\frac{\partial}{\partial\chi} \tag{2.3}$$

where we also introduced the operator v_3 which is needed later. The $SO(2,1)$ invariant line element $\mathrm{d}s^2 = -\mathrm{d}X_1^2 - \mathrm{d}X_2^2 + \mathrm{d}X_3^2$ in this case is

$$\mathrm{d}s^2 = \frac{4}{(1-R^2)^2}\,\mathrm{d}R^2 - \left(\frac{1+R^2}{1-R^2}\right)^2\mathrm{d}\chi^2 = g_{\mu\nu}\,\mathrm{d}y^\mu\,\mathrm{d}y^\nu \tag{2.4}$$

where $(y_1, y_2) \equiv (R, \chi)$. The generators (2.1) in terms of our new coordinates can be expressed as

$$L_1 = Kv_2 + \frac{2}{1+R^2}Rv_1v_3 \tag{2.5a}$$

$$L_2 = -Kv_1 + \frac{2}{1+R^2}Rv_2v_3 \qquad L_3 = v_3 \tag{2.5b}$$

with $K = \frac{1}{2}(1-R^2)P$, $P \equiv -\mathrm{i}\frac{\partial}{\partial R}$

Using the (2.5) realization now we calculate the Casimir operator C for $SO(2,1)$. The result is

$$C(SO(2,1)) = -{L_1}^2 - {L_2}^2 + {L_3}^2 = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu). \tag{2.6}$$

or explicitly after using a similarity transformation $T(R) = \sqrt{\frac{1-R^2}{1+R^2}}$

$$T^{-1}C(SO(2,1))T = \frac{1}{4}(1-R^2)\partial_R(1-R^2)\partial_R + \left(v_3^2 - \frac{1}{4}\right)\left(\frac{1-R^2}{1+R^2}\right)^2 - \frac{1}{4}. \tag{2.7}$$

Notice that the operator $K \equiv \frac{1}{2}(1-R^2)P$ generates an $SO(1,1)$ group, and the first term of the Casimir is simply $-K^2$ which can be regarded as an $SO(1,1)$ Casimir operator.

After employing the variable transformation $R(r) = \tanh\frac{r}{2}$ we get

$$T^{-1}C(SO(2,1))T = {\partial_r}^2 + \frac{v_3^2 - \frac{1}{4}}{\cosh r^2} - \frac{1}{4}. \tag{2.8}$$

Acting with this Casimir operator on the states $|j, v\rangle$ where $v_3|j, v\rangle = v|j, v\rangle$ and $C|j, v\rangle = j(j+1)|j, v\rangle$, and choosing the principal series of irreducible representations with $j = -\frac{1}{2}+\mathrm{i}k$ we get a Schrödinger equation for a *one-dimensional* scattering problem with the potential $-(v^2 - \frac{1}{4})/\cosh r^2$ which reproduces the well known result in [10].

2.2. Non-trivial inducing representation

However, we are more ambitious and we are interested in the cases where $q \neq 0$, i.e. the inducing $SO(1, 1)$ representation is non-trivial. The task is now to find suitable q-dependent modifications to the operators of (2.5). In other words, we have to find the infinitesimal generators of the induced representation for $SO(2, 1)$, induced by a representation of $SO(1, 1)$ labelled by q. Based on the results of [3] the new modified generators satisfying the $SO(2, 1)$ commutation relations are

$$J_1 = L_1 + q\frac{1 - R^2}{1 + R^2}v_1 \qquad J_2 = L_2 + q\frac{1 - R^2}{1 + R^2}v_2 \qquad J_3 = L_3. \tag{2.9a}$$

The explicit construction of these generators is similar to the one presented in [3] for the group $SO(3)$ using the polar coordinates (θ, ϕ) for the coset $SO(3)/SO(2)$ (the two-sphere). To obtain the $SO(2, 1)/SO(1, 1)$ case we employ the similarity transformation $\mathrm{e}^{\mathrm{i}n\phi}$ (gauge transformation) to the $SO(3)$ generators. By using the identification $n \equiv q$, $(\theta, \phi) \equiv (\mathrm{i}r+\frac{\pi}{2}, \chi)$ in the relevant formulae of [3] we obtain the generators of (2.9*a*). Of course the minimal form of modification (2.9*a*) can easily be guessed without any recourse to this construction. A further discussion on the properties of the modification added to $\mathbf{L}$ is given in appendix A. Moreover, for later purposes we rewrite (2.9*a*) as

$$J_1 = Kv_2 + fRv_1(v_3 - qR) + qv_1 \qquad J_2 = -Kv_1 + fRv_2(v_3 - qR) + qv_2 \qquad J_3 = L_3. \tag{2.9b}$$

where $f(R) \equiv \frac{2}{1+R^2}$.

We can now calculate the Casimir operator $\mathcal{C}$ for our new realization. For the $q = 0$ case we indicated that the quadratic Casimir $C(SO(2, 1))$ can be written as the Laplace–Beltrami operator for an appropriately chosen metric. (See equations (2.4) and (2.6).) Based on this result it is an interesting question whether it is possible to identify a similar structure for the modified Casimir $\mathcal{C}(SO(2, 1))$. Indeed one can prove that a formula similar to (2.6) is true for $\mathcal{C}(SO(2, 1))$. It is

$$\mathcal{C}(SO(2, 1) = -{J_1}^2 - {J_2}^2 + {J_3}^2 = \frac{1}{\sqrt{g}}(\partial_\mu + \mathrm{i}A_\mu)(\sqrt{g}g^{\mu\nu}(\partial_\nu + \mathrm{i}A_\nu)) + q^2 \tag{2.10}$$

where

$$A_\mu = \begin{pmatrix} A_R \\ A_\chi \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{2qR}{R^2 - 1} \end{pmatrix}. \tag{2.11}$$

It is shown in appendix A that $\mathbf{A}$ acts like a vector potential, i.e. an $so(1, 1)$-valued gauge field. Moreover, using the variable r instead of R the one-form $A = A_R\,\mathrm{d}R + A_\chi\,\mathrm{d}\chi = -q\sinh r\,\mathrm{d}\chi$ is a monopole-like gauge field with pole strength q.

Our next step is to derive the potential from our new realization. Using the same steps as for the $q = 0$ case we get the potential

$$V(r) = -\frac{v^2 - q^2 + 2qv\sinh r - \frac{1}{4}}{\cosh r^2} \tag{2.12}$$

which is the so-called Gendenshtein potential discussed in [11]. Notice that q can also be complex, provided we chose a non-unitary representation for the inducing $SO(1, 1)$ subgroup.

Hence the (2.12) potential is a complex (optical) potential. In order to accommodate properly this property of the interaction term we also have to choose the $SO(2,1)$ representation content appropriately. Choosing a non-unitary representation with $j = -\frac{1}{2} + \mathrm{i}k$ with $k \in \mathbb{C}$, we will have a complex number for $k^2 = E$. Solvable complex potentials have already been investigated in the literature [12]. The use of non-unitary representations for non-compact groups appeared first in [13]. Moreover, the reflection coefficient (S-matrix) can be calculated [11]; we merely give the result:

$$R(k, v, q) = \left(\frac{\cos(v - 1/2)\sinh \pi q}{\cosh \pi k} + \mathrm{i}\frac{\sin(v - 1/2)\cosh \pi q}{\sinh \pi k} \right) T(k, v, q) \tag{2.13a}$$

where

$$T(k, v, q) = \frac{\Gamma(1/2 - v - \mathrm{i}k)\Gamma(1/2 + v - \mathrm{i}k)\Gamma(1/2 + \mathrm{i}q - \mathrm{i}k)\Gamma(1/2 - \mathrm{i}q - \mathrm{i}k)}{\Gamma(-\mathrm{i}k)\Gamma(1 + \mathrm{i}k)\Gamma^2(1/2 - \mathrm{i}k)}. \tag{2.13b}$$

3. A realization for $SO(2, 2)$ using coordinates on the coset $SO(2, 2)/SO(2, 1)$

Our next step is to repeat the constructions of the previous section for a non-compact group with a non-compact *non-Abelian* subgroup. In this case we have the possibility of having a *finite-dimensional non-unitary matrix representation* $\mathcal{D}$ as the inducing representation. The simplest example of this kind is the choice $G = SO(2, 2)$ and $H = SO(2, 1)$. We choose a finite-dimensional matrix representation for the generators of $H = SO(2, 1)$. Since this representation is non-unitary some of the generators S_k, $k = 1, 2, 3$ are *non-Hermitian matrices* satisfying the usual $SO(2, 1)$ commutation relation, i.e. we have

$$[S_1, S_2] = -\mathrm{i}S_3 \qquad [S_2, S_3] = \mathrm{i}S_1 \qquad [S_3, S_1] = \mathrm{i}S_2. \tag{3.1}$$

For later use we rename these generators as follows: $(S_1, S_2, S_3) \equiv (T_1, T_2, S)$.

3.1. Trivial inducing representation

We represent the three-dimensional coset space $SO(2, 2)/SO(2, 1)$ as the set $\mathcal{M} = \{X_a,\ a = 1, 2, 3, 4 | -X_1^2 - X_2^2 + X_3^2 + X_4^2 = 1\}$ [9]. The generators of $SO(2, 2)$ are differential operators corresponding to the action of $SO(2, 2)$ on $\mathcal{M}$. These generators are

$$M_1 = X_2P_3 + X_3P_2 \qquad M_2 = -X_1P_3 - X_3P_1 \qquad M_3 = X_1P_2 - X_2P_1 \tag{3.2a}$$

$$N_1 = X_1P_4 + X_4P_1 \qquad N_2 = X_2P_4 + X_4P_2 \qquad N_3 = X_3P_4 - X_4P_3 \tag{3.2b}$$

where $P_a = -\mathrm{i}\partial_a$, $a = 1, \ldots, 4$. The commutation relations are

$$[M_1, M_2] = -\mathrm{i}M_3 \qquad [M_2, M_3] = \mathrm{i}M_1 \qquad [M_3, M_1] = \mathrm{i}M_2 \tag{3.3a}$$

$$[M_1, N_2] = -\mathrm{i}N_3 \qquad [M_2, N_3] = \mathrm{i}N_1 \qquad [M_3, N_1] = \mathrm{i}N_2 \tag{3.3b}$$

$$[N_1, M_2] = -\mathrm{i}N_3 \qquad [N_2, M_3] = \mathrm{i}N_1 \qquad [N_3, M_1] = \mathrm{i}N_2$$

$$[N_1, N_2] = -\mathrm{i}M_3 \qquad [N_2, N_3] = \mathrm{i}M_1 \qquad [N_3, N_1] = \mathrm{i}M_2. \tag{3.3c}$$

Since we have a three-dimensional coset we introduce the coordinates y_μ, $\mu = 1, 2, 3$, $(y_1, y_2, y_3) = (R_1, R_2, \chi)$ by

$$X_1 = \frac{2}{1 - R^2}R_1 \qquad X_2 = \frac{2}{1 - R^2}R_2 \qquad X_3 = \frac{1 + R^2}{1 - R^2}v_1 \qquad X_4 = \frac{1 + R^2}{1 - R^2}v_2 \tag{3.4}$$

with v_1, v_2 defined as in (2.3). The use of the coordinates R_1, R_2 shows that unlike in the previous case (one-dimensional scattering problem with R being the relevant coordinate) we expect a *two-dimensional* scattering problem. This fact is also suggested by the group chain $SO(2) \subset SO(2,1) \subset SO(2,2)$ where $SO(2)$ can be regarded as angular momentum in two dimensions.

The line element in these coordinates is

$$ds^2 = \frac{4}{(1-R^2)^2}\, d\mathbf{R}^2 - \left(\frac{1+R^2}{1-R^2}\right)^2 d\chi^2 = g_{\mu\nu}\, dy^\mu\, dy^\nu \tag{3.5}$$

where $R^2 = R_1^2 + R_2^2$. In terms of the coordinates (R_1, R_2, χ) the $so(2,2)$ generators have the form

$$\mathbf{M} = \mathbf{K}^\star v_1 - \frac{2}{1+R^2}\mathbf{R}^\star v_2 v_3 \qquad M_3 = L \tag{3.6a}$$

$$\mathbf{N} = \mathbf{K} v_2 + \frac{2}{1+R^2}\mathbf{R} v_1 v_3 \qquad N_3 = v_3 \tag{3.6b}$$

with

$$\mathbf{K} = \frac{1}{2}(1+R^2)\mathbf{P} - \mathbf{R}(\mathbf{RP}) \qquad L = R_1 P_2 - R_2 P_1 \tag{3.7}$$

where $\mathbf{V} = (V_1, V_2)$ denotes a two-component vector of operators and $\mathbf{V}^\star = (V_2, -V_1)$ is its *dual*, i.e. $V_I^\star = \varepsilon_{IJ} V_J$, with $\varepsilon_{12} = -\varepsilon_{21} = 1$, $\mathbf{P} = (P_1, P_2) = (-i\partial/\partial R_1, -i\partial/\partial R_2)$.

It is important to notice at this point that the operators $(\mathbf{K}^\star, L)$ generate an $SO(2,1)$ algebra. These operators generate the infinitesimal action of $SO(2,1)$ on the coset $SO(2,1)/SO(2)$ which is the *double-sheet hyperboloid* $-X_1^2 - X_2^2 + Z^2 = 1$ parametrized as $\mathbf{X} = 2\mathbf{R}/(1-R^2)$, $Z = (1+R^2)/(1-R^2)$. It is instructive to compare this with the results obtained for the *one-sheet hyperboloid*, equations (2.2) and (2.5). One can say that our $SO(2,2)$ algebra is built up by using the generators $(\mathbf{K}^\star, L)$ of an $SO(2,1)$ algebra.

Using the realization (3.6) one can calculate the Casimir operators

$$C(SO(2,2)) = -M_1^2 - M_2^2 + M_3^2 - N_1^2 - N_2^2 + N_3^2 \tag{3.8a}$$

$$C'(SO(2,2)) = -M_1 N_1 - M_2 N_2 + M_3 N_3. \tag{3.8b}$$

Notice that $so(2,2)$ is a rank-two Lie algebra, hence it has *two* independent Casimir operators. However, for the realization (3.6) a calculation shows that $C' = 0$, and C can be expressed in the form (2.6) in terms of the Laplace–Beltrami operator of $\mathcal{M}$ (now $\mu, \nu = 1, 2, 3$ and the metric $g_{\mu\nu}$ is obtained from (3.5)). Explicitly we have

$$C(SO(2,2)) = C(SO(2,1)) + \frac{1-R^2}{1+R^2} R\partial_R + \left(\frac{1-R^2}{1+R^2}\right)^2 v_3^2 \tag{3.9a}$$

$$C(SO(2,1)) = \frac{1}{4}(1-R^2)^2\left(\partial_R^2 + \frac{1}{R}\partial_R - \frac{L^2}{R^2}\right). \tag{3.9b}$$

We notice that the $SO(2,1)$ Casimir operator $C(SO(2,1) = -\mathbf{K}^2 + L^2$ appears in the $SO(2,2)$ Casimir. This can be traced back to the $so(2,1)$ algebraic structure sitting inside our $so(2,2)$ algebra. (See the presence of $\mathbf{K}$ and L in (3.6).)

Using the same similarity transformation as in (2.7) we get

$$T^{-1}C(SO(2,2))T = C(SO(2,1)) + (v_3^2 - \tfrac{1}{4})\left(\frac{1-R^2}{1+R^2}\right)^2 - \frac{3}{4}. \tag{3.10}$$

After the usual change of variable $R(r) = \tanh \frac{r}{2}$ we get

$$T^{-1}C(SO(2,2))T = \partial_r{}^2 + \frac{v_3{}^2 - \frac{1}{4}}{\cosh^2 \frac{r}{2}} - \frac{L^2 - \frac{1}{4}}{\sinh^2 \frac{r}{2}} - 1. \tag{3.11}$$

Acting with this Casimir operator on the states $|\omega, m, v\rangle$ where $v_3|\omega, m, v\rangle = v|\omega, m, v\rangle$, $L|\omega, m, v\rangle = m|\omega, m, v\rangle$, and $C|\omega, m, v\rangle = \omega(\omega + 2)|\omega, m, v\rangle$, and choosing the principal series of irreducible representations with $\omega = -1 + \mathrm{i}k$ we get a Schrödinger equation for a *two-dimensional* scattering problem with the potential

$$-\frac{m^2}{r^2} + \frac{m^2 - \frac{1}{4}}{\sinh^2 \frac{r}{2}} - \frac{v^2 - \frac{1}{4}}{\cosh^2 \frac{r}{2}}$$

in agreement with a similar result in [14].

3.2. Non-trivial inducing representations

Now we come to the point of modifying the generators (3.6), by adding matrix-valued modifications to them. For this purpose we use the finite-dimensional (non-Hermitian) matrix realization (3.1). Our construction is based on the observation that *an $so(2, 1)$ algebra sits inside the realization (3.6) which is realized in terms of coordinates on the coset* $SO(2, 1)/SO(2)$. Indeed, we know how to modify $\mathbf{K}$ and L from [4]. The result in terms of our coordinates (R_1, R_2) is

$$J = L + S \qquad \mathbf{I} = \mathbf{K} - \mathbf{R}^\star S \tag{3.12}$$

where we take the matrix S to be equal to S_3 of (3.1). One can check that the $so(2, 1)$ commutation relations are satisfied. Now we search for a modification of the realization in (3.6) by replacing $\mathbf{K}$ and L by $\mathbf{I}$ and J and adding suitable overall modifications to $\mathbf{M}$ and $\mathbf{N}$. *N_3 is not modified.*

It is instructive at this point to recall the results of the previous chapter. Indeed this $SO(2, 1)/SO(1, 1)$ case will be the archetypical example for what follows. According to (2.5) and (2.9) the generator $L_3 = v_3$ *was not modified*. Moreover, the generator of an $SO(1, 1)$ algebra K was also present in (2.5). There we did not have to modify the term Kv_2 (Kv_1), since there was no proper compact Lie subgroup of $SO(1, 1)$ to start the inducing construction with. The modification we used there (containing the multiplying factors v_1 (v_2) was written in the form (2.9*b*).

If we follow this prescription, it is sensible to employ the choice

$$\mathcal{M} = \mathbf{I}^\star v_1 - f(R)\mathbf{R}^\star v_2(v_3 - \mathbf{TR}) - \mathbf{T}^\star v_2 \qquad \mathcal{M}_3 = J = L + S \tag{3.13a}$$

$$\mathcal{N} = \mathbf{I}v_2 + f(R)\mathbf{R}v_1(v_3 - \mathbf{TR}) + \mathbf{T}v_1 \qquad \mathcal{N}_3 = v_3 \tag{3.13b}$$

where $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2)$, $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2)$, and $f(R) = \frac{2}{1+R^2}$ as usual. A straightforward but tedious calculation shows that these generators indeed satisfy the (3.3) commutation relations, hence they form a matrix-valued realization of the $so(2, 2)$ algebra. Moreover, by construction this representation is the induced representation induced by the non-unitary matrix representation of the $SO(2, 1)$ subgroup.

3.3. The scattering problem for a special choice of the inducing representation

Now the next step is the calculation of the Casimir operators for our realization (3.13), i.e.

$$\mathcal{C}(SO(2,2)) = -\mathcal{M}_1^2 - \mathcal{M}_2^2 + \mathcal{M}_3^2 - \mathcal{N}_1^2 - \mathcal{N}_2^2 + \mathcal{N}_3^2 \tag{3.14a}$$

$$\mathcal{C}'(SO(2,2)) = -\mathcal{M}_1\mathcal{N}_1 - \mathcal{M}_2\mathcal{N}_2 + \mathcal{M}_3\mathcal{N}_3. \tag{3.14b}$$

$$\mathcal{C}(SO(2,2)) = \mathcal{C}(SO(2,1)) + \frac{1-R^2}{1+R^2} R\partial_R + 2f(\mathbf{RT}(v_3 - \mathbf{RT})) + (Rf)^2(v_3 - \mathbf{RT})^2 \tag{3.15a}$$

$$\mathcal{C}'(SO(2,2)) = K_1T_2 - K_2T_1 + L(\mathbf{RT}) + \frac{1-R^2}{1+R^2} S(v_3 - \mathbf{RT}) \tag{3.15b}$$

where

$$\mathcal{C}(SO(2,1)) = -I_1^2 - I_2^2 + J^2 = C(SO(2,1)) + \tfrac{1}{2}(1-R^2)(1/2 + 2LS) \tag{3.16}$$

and we used equations (3.7), (3.9*b*) and (3.12) with $f(R) = \frac{2}{1+R^2}$.

In this paper we specialize merely to the simplest non-Hermitian realization of (3.1), hence we take

$$T_1 = -\frac{\mathrm{i}}{2}\sigma_2 \qquad T_2 = \frac{\mathrm{i}}{2}\sigma_1 \qquad S = \frac{1}{2}\sigma_3. \tag{3.17}$$

Employing (3.17), and then using the usual similarity transformation (2.7) we get

$$T^{-1}\mathcal{C}(SO(2,2))T = \mathcal{C}(SO(2,1)) - 2\sigma_3(\boldsymbol{\sigma} R)\frac{1-R^2}{1+R^{2^2}}v_3 + \left(\frac{1-R^2}{1+R^2}\right)^2 v_3^2 - \frac{1}{2} \tag{3.18a}$$

$$T^{-1}\mathcal{C}'(SO(2,2))T = \frac{\mathrm{i}}{4}(1-R^2)(\boldsymbol{\sigma} P) + \frac{1}{4}(\boldsymbol{\sigma} R) + \frac{1}{2}\sigma_3 v_3 \frac{1-R^2}{1+R^2} \tag{3.18b}$$

where $\boldsymbol{\sigma} P = \sigma_1 P_1 + \sigma_2 P_2$, and $\boldsymbol{\sigma} R = \sigma_1 R_1 + \sigma_2 R_2$.

In the realization (3.17) there exists an important relation between $\mathcal{C}$ and $\mathcal{C}'$, namely

$$(2\mathcal{C}'(SO(2,2)))^2 = \mathcal{C}(SO(2,2)) + \tfrac{3}{4} \tag{3.19}$$

as can be proved using the similar formula

$$(2D(SO(2,1)))^2 = \mathcal{C}(SO(2,1)) + \tfrac{1}{4} \tag{3.20}$$

where $D \equiv \frac{1}{4}\mathrm{i}(1-R^2)(\boldsymbol{\sigma} P) + \frac{1}{4}(\boldsymbol{\sigma} R)$.

Now, following [6], we show that (3.20) fixes the representation content of the scattering states. The irreducible representations of $SO(2,2)$ capable of characterizing scattering states are classified by the pair (j_0, j_1), where $j_0 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and $j_1 = \mathrm{i}k,\ k \in \mathbf{R}_0^+$. According to AST scattering states are labelled as $|j_0, j_1\rangle$. The action of the Casimir operators on this base [9] is

$$\mathcal{C}|j_0, j_1\rangle = ({j_0}^2 + {j_1}^2 - 1)|j_0, j_1\rangle \tag{3.21a}$$

$$\mathcal{C}'|j_0, j_1\rangle = j_0 j_1 |j_0, j_1\rangle. \tag{3.21b}$$

Using equations (3.19) we get the relation $(j_0^2 - \frac{1}{4})(j_1^2 - \frac{1}{4}) = 0$ so we can single out the states $(\pm\frac{1}{2}, \mathrm{i}k)$. Moreover relation (3.19) is very important since we merely have to deal with a *first-order differential operator* when we are trying to simplify our problem. Our Hamiltonian (which turns out to be $\mathcal{C} + \frac{3}{4}$) is simply four times the square of this operator.

Let us try to simplify (3.18*b*). Using the coordinate transformation $R(r) = \tanh\frac{r}{2}$ followed by a further similarity transformation $S(r) = (\sinh r)^{-1/2}$ we get

$$2(TS)^{-1}\mathcal{C}'(SO(2,2))(TS) = \boldsymbol{\sigma} n\left(\partial_r - \frac{\mathcal{K}}{\sinh r}\right) + \sigma_3 \frac{v_3}{\cosh r} \tag{3.22}$$

where $\mathcal{K} = \frac{1}{2} + \sigma_3 L$, $\mathbf{n} = \mathbf{R}/R$ is the unit vector in the $\mathbf{R}$ direction. Important relations satisfied by these quantities are $\{\mathcal{K}, \boldsymbol{\sigma} n\} = 0$, $\{\sigma_3, \boldsymbol{\sigma} n\} = 0$ where $\{\,,\}$ is the anticommutator. Using these relations one can quickly see that $\mathcal{C}'$ is *not* Hermitian, hence k is a complex number.

Since the operators J and v_3 are commuting with both of the Casimir operators and among themselves we can characterize the scattering states (3.21) further as

$$J|1/2,k,j,v,\pm\rangle = j|1/2,k,j,v,\pm\rangle \qquad v_3|1/2,k,j,v,\pm\rangle = v|1/2,k,j,v,\pm\rangle \tag{3.23}$$

where $\pm$ refers to the two-component nature of the state vector corresponding to the cases $m = j \mp 1/2$, respectively. The action of the operator $\mathcal{M}$ on these states is

$$\mathcal{K}|1/2,k,j,v,\pm\rangle = \pm j|1/2,k,j,v,\pm\rangle. \tag{3.24}$$

Acting with (3.22) on

$$\psi_{1/2,k,j,v,\pm}(r,\varphi,\chi) \equiv \langle r,\varphi,\chi|1/2,k,j,v,\pm\rangle = \begin{pmatrix} F_{k,j,v,+}(r)\mathrm{e}^{\mathrm{i}(j-\frac{1}{2})\varphi}\mathrm{e}^{\mathrm{i}v\chi} \\ F_{k,j,v,-}(r)\mathrm{e}^{\mathrm{i}(j+\frac{1}{2})\varphi}\mathrm{e}^{\mathrm{i}v\chi} \end{pmatrix} \tag{3.25}$$

one can easily check that equations (3.23) and (3.24) are satisfied. Moreover, according to (3.21*b*) we have to also satisfy equation $2(TS)^{-1}\mathcal{C}'TS\psi_{k,\ldots} = \mathrm{i}k\psi_{k,\ldots}$, which by virtue of (3.22) yields the following set of radial equations for the unknown functions $F_\pm(r)$:

$$\begin{pmatrix} \dfrac{v}{\cosh r} & \partial_r + \dfrac{j}{\sinh r} \\ \partial_r - \dfrac{j}{\sinh r} & -\dfrac{v}{\cosh r} \end{pmatrix} \begin{pmatrix} F_{k,j,v,+}(r) \\ F_{k,j,v,-}(r) \end{pmatrix} = \mathrm{i}k \begin{pmatrix} F_{k,j,v,+}(r) \\ F_{k,j,v,-}(r) \end{pmatrix} \tag{3.26}$$

where we have used the special form $\sigma_1\cos\varphi + \sigma_2\sin\varphi$ of the operator $\boldsymbol{\sigma}\mathbf{n}$ when acting on the corresponding states.

Using the special form of the matrix $2(TS)^{-1}\mathcal{C}'(TS)$ on the left-hand side of (3.26), we can easily calculate its square, which according to (3.19) is just $\mathcal{C} + \frac{3}{4}$ with the eigenvalue $-k^2$. Since it is $-1\times$ the scattering energy, $-(\mathcal{C} + \frac{3}{4})$ is just the scattering Hamiltonian. Calculating the square and employing a further similarity transformation $Q(r) = 1/\sqrt{r}$, we obtain the following radial Schrödinger equation:

$$\left[\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{m^2}{r^2} + k^2\right)\delta_{\alpha\beta} - V^{j,v}_{\alpha\beta}(r)\right]\frac{1}{\sqrt{r}}F_{k,j,v,\alpha} = 0 \tag{3.27}$$

with the matrix-valued potential

$$V^{j,v}(r) = -\frac{m^2 - 1/4}{r^2} + \frac{j^2 + j\sigma_3\cosh r}{\sinh^2 r} - \frac{v^2 - \mathrm{i}v\sigma_2\sinh r}{\cosh^2 r} \tag{3.28}$$

where the matrix indicies $\alpha \in \{\pm\}$ correspond to the channels $j = m \mp 1/2$. The interaction term obtained is non-Hermitian, hence it represents an optical potential. Moreover, as discussed previously, k^2 is complex.

4. Solutions of the scattering problem in special cases

Our task is to solve the Schrödinger equation (3.27) with the non-Hermitian interaction term (3.28). In order to do this we use the first-order equation (3.26) which is easier to handle. Unfortunately we did not manage to find the general set of solutions for the whole two-parameter family of potentials. We merely present results for the special cases $(j,v) \equiv (j,0)$, and $(j,v) \equiv (0,v)$. Since $j = m \pm 1/2$ $m \in \mathbb{Z}$, for the second choice ($j = 0$) we have to use representations of the $so(2,2)$ algebra wich are double-valued representations of the group $SO(2,2)$ [10].

4.1. The case with $v = 0$

For our first choice we set $v = 0$ in (3.26). The resulting coupled set of first-order equations yields eigenfunctions of the Schrödinger equation (3.27) with the potential containing the term $\frac{j^2 \pm j \cosh r}{\sinh^2 r}$, which is a well known example of a solvable potential. However, for later purposes it is instructive to present an alternative solution to this problem using merely (3.26) which is now of the form

$$\begin{pmatrix} 0 & \partial_r + \dfrac{j}{\sinh r} \\ \partial_r - \dfrac{j}{\sinh r} & 0 \end{pmatrix} \begin{pmatrix} F_{k,j,0,+}(r) \\ F_{k,j,0,-}(r) \end{pmatrix} = \mathrm{i}k \begin{pmatrix} F_{k,j,0,+}(r) \\ F_{k,j,0,-}(r) \end{pmatrix}. \tag{4.1}$$

Since the operator on the left-hand side is anti-Hermitian, its eigenvalue is purely imaginary, hence $k \in \mathbb{R}$. We write the unknown functions $F_\pm$ in the form

$$F_{k,j,0,\pm}(r) = \sqrt{\sinh r}\,\mathcal{F}_{k,j,0,\pm}(r) \tag{4.2}$$

and change the variables as $z(r) = \cosh r$. The resulting coupled set of equations is

$$\begin{pmatrix} 0 & \sqrt{z^2-1}\partial_z + \dfrac{z/2+j}{\sqrt{z^2-1}} \\ \sqrt{z^2-1}\partial_z + \dfrac{z/2-j}{\sqrt{z^2-1}} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{F}_{k,j,0,+}(z) \\ \mathcal{F}_{k,j,0,-}(z) \end{pmatrix} = \mathrm{i}k \begin{pmatrix} \mathcal{F}_{k,j,0,+}(z) \\ \mathcal{F}_{k,j,0,-}(z) \end{pmatrix}. \tag{4.3}$$

These are precisely the equations [15] for the functions $\mathcal{B}^l_{mn}(z)$ which are the generalizations of Wigner's d-functions for the group $SU(1,1)$, with the choice $l = -1/2 + \mathrm{i}k$, $m = -j$, $n = \pm 1/2$, i.e. using equation (4.2) we have

$$F_{k,j,0,\pm}(r) = \sqrt{\sinh r}\,\mathcal{B}^{-\frac{1}{2}+\mathrm{i}k}_{-j,\mp\frac{1}{2}}(\cosh r). \tag{4.4}$$

Moreover, an analogous set of equations [16] coupling the functions $\mathcal{B}^l_{mn}(z)$ and $\mathcal{B}^l_{m\pm 1n}(z)$ can also be used to show that the functions

$$F'_{k,j,0,\pm}(r) = \sqrt{\sinh r}\,\mathcal{B}^{-\frac{1}{2}+\mathrm{i}k}_{\pm\frac{1}{2}j}(\cosh r) \tag{4.5}$$

also satisfy (4.3). According to [15] the functions $\mathcal{B}^l_{mn}(z)$ can be expressed in terms of the hypergeometric function ${}_2F_1$ in the following form:

$$\mathcal{B}^l_{mn}(z) = \frac{(z-1)^{\frac{(n-m)}{2}}(z+1)^{\frac{(n+m)}{2}}}{2^n\Gamma(n-m+1)}\,{}_2F_1\left(l+n+1, n-l; n-m+1; \tfrac{1}{2}(1-z)\right). \tag{4.6}$$

According to this result and (4.4), the wavefunctions that are regular at the origin are

$$F_{k,j,0,-}(r) = \frac{2^j}{\Gamma(\frac{3}{2}+j)} \sinh r \left(\tanh\frac{r}{2}\right)^j {}_2F_1\left(1+\mathrm{i}k, 1-\mathrm{i}k, \tfrac{3}{2}+j, -\sinh^2\tfrac{r}{2}\right) \tag{4.7a}$$

$$F_{k,j,0,+}(r) = \frac{2^j}{\Gamma(\frac{1}{2}+j)} \left(\tanh\frac{r}{2}\right)^j {}_2F_1\left(\mathrm{i}k, -\mathrm{i}k, \tfrac{1}{2}+j, -\sinh^2\tfrac{r}{2}\right) \tag{4.7b}$$

where we have chosen $j \geqslant 1$.

Using the asymptotic property of the hypergeometric function [17]

$$\lim_{|z|\to\infty} {}_2F_1(a,b,c;z) = \Gamma(c)\left(\frac{\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} + \frac{\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\right) \tag{4.8}$$

one can verify that

$$\lim_{r\to\infty} F_{k,j,0,\pm}(r) \sim \mathrm{e}^{-\mathrm{i}kr} \mp 2^{-4\mathrm{i}k} \frac{\Gamma(2\mathrm{i}k)}{\Gamma(-2\mathrm{i}k)} \frac{\Gamma(-\mathrm{i}k)}{\Gamma(\mathrm{i}k)} \frac{\Gamma(1/2+j-\mathrm{i}k)}{\Gamma(1/2+j+\mathrm{i}k)} \mathrm{e}^{\mathrm{i}kr}. \tag{4.9}$$

After using the property $\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1/2)$ of gamma functions, the S-matrix (having only diagonal elements in this case) is

$$S_{j,0,\pm}(k) = \pm \frac{\Gamma(1/2+\mathrm{i}k)}{\Gamma(1/2-\mathrm{i}k)} \frac{\Gamma(1/2+j-\mathrm{i}k)}{\Gamma(1/2+j+\mathrm{i}k)}. \tag{4.10}$$

4.2. The case with $j = 0$

Now we turn to the other special case, namely that with $(j, v) \equiv (0, v)$. As we have already noticed, the values of the parameters j, m, v in the interaction term (3.28) can be continued to arbitrary real values. Such representations of the Lie algebra correspond to multivalued reresentations of the Lie group. Putting $j = 0$ in (3.26) we get the equation

$$\begin{pmatrix} \dfrac{v}{\cosh r} & \partial_r \\ \partial_r & -\dfrac{v}{\cosh r} \end{pmatrix} \begin{pmatrix} F_{k,0,v,+}(r) \\ F_{k,0,v,-}(r) \end{pmatrix} = \mathrm{i}k \begin{pmatrix} F_{k,0,v,+}(r) \\ F_{k,0,v,-}(r) \end{pmatrix}. \tag{4.11}$$

Here the matrix-valued differential operator is *not* Hermitian: hence the eigenvalue k is *complex*. In order to solve (4.11) we employ the variable transformation $\theta = \mathrm{i}r + \pi/2$, hence $\cos\theta = -\mathrm{i}\sinh r$ and $\sin\theta = \cosh r$, $\partial/\partial r = \mathrm{i}\partial/\partial\theta$. Using the linear combinations

$$G_\pm = F_- \pm \mathrm{i}F_+ \tag{4.12}$$

equation (4.10) has the form

$$\begin{pmatrix} 0 & -\partial_\theta - \dfrac{v}{\sin\theta} \\ \partial_\theta - \dfrac{v}{\sin\theta} & 0 \end{pmatrix} \begin{pmatrix} G_{k,0,v,+}(\theta) \\ G_{k,0,v,-}(\theta) \end{pmatrix} = \mathrm{i}k \begin{pmatrix} G_{k,0,v,+}(\theta) \\ G_{k,0,v,-}(\theta) \end{pmatrix}. \tag{4.13}$$

Notice that the transformation (4.12) diagonalizes the interaction term

$$V(r) = -\frac{v^2 - \mathrm{i}v\sigma_2 \sinh r}{\cosh^2 r}$$

the only term surviving after putting $j = 0$ $(m = \pm 1/2)$ in (3.28). In this basis $V(r)$ has the form

$$V(r) = -\frac{v^2 \pm \mathrm{i}v \sinh r}{\cosh^2 r} \tag{4.14}$$

which is just the Gendenshtein potential of (2.12) with $q = \pm\frac{\mathrm{i}}{2}$. However, in section 2 we had $-\infty < r + \infty$; now $0 \leqslant r < +\infty$ where r is a radial coordinate. Hence it follows that we also have to satisfy the boundary conditions $G_\pm(0) = 0$. Moreover, just as in the $(j, v) = (j, 0)$ case the potentials of (4.14) are supersymmetric partners of one another, hence the S-matrices (just as in equation (4.10)) should be the same up to a sign [11].

In order to find the solutions we introduce the variable $w(\theta) = \cos\theta$ and define

$$G_{k,0,v,\pm}(r) = \sqrt{\sin\theta}\,\mathcal{G}_{k,0,v,\pm}(r) \tag{4.15}$$

to obtain the equation

$$\begin{pmatrix} 0 & \sqrt{1-w^2}\,\partial_w + \dfrac{-w/2 - v}{\sqrt{1-w^2}} \\ -\sqrt{1-w^2}\,\partial_w + \dfrac{w/2 - v}{\sqrt{1-w^2}} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{G}_+(w) \\ \mathcal{G}_-(w) \end{pmatrix} = \mathrm{i}k \begin{pmatrix} \mathcal{G}_+(w) \\ \mathcal{G}_-(w) \end{pmatrix} \tag{4.16}$$

which is similar to (4.3). According to [15] the solutions to these equations can be expressed in terms of the functions $P^l_{mn}(w)$. Using equation (4.15) it is easy to see that the solutions are

$$G_{k,0,v,\pm}(r) = \sqrt{\sin\theta}\,P^{-\frac{1}{2}+\mathrm{i}k}_{v,\pm\frac{1}{2}}(\cos\theta) = \sqrt{\cosh r}\,P^{-\frac{1}{2}+\mathrm{i}k}_{v,\pm\frac{1}{2}}(-\mathrm{i}\sinh r) \qquad (4.17)$$

where k is a complex number. The other pair of solutions (see (4.5) in this respect) is just given by using $P^{-\frac{1}{2}+\mathrm{i}k}_{\pm\frac{1}{2},v}(-\mathrm{i}\sinh r)$ in equation (4.17). Using the properties of the P^l_{mn} functions expressed in terms of the hypergeometric function, (a formula similar to (4.6) is valid), it is not hard to see that our wavefunctions are linear combinations of the two linearly independent solutions, i.e.

$$G_{k,v,\pm}(r) = E_{v,\pm}(r)\Big(c_1 F\big(1/2+v+\mathrm{i}k, 1/2+v-\mathrm{i}k, 1+v\pm 1/2, z(r)\big)$$
$$+\, c_2 z(r)^{-v\mp\frac{1}{2}} F\big(1/2\mp 1/2+\mathrm{i}k, 1/2\mp 1/2-\mathrm{i}k, 1-v\mp 1/2, z(r)\big)\Big) \qquad (4.18)$$

where

$$z(r) = \tfrac{1}{2}(1+\mathrm{i}\sinh r) \qquad E_{v,\pm} = (\cosh r)^{\frac{1}{2}+v}\exp\big(\pm\mathrm{i}\tfrac{1}{2}\tan^{-1}\sinh r\big). \qquad (4.19)$$

We would like to choose the complex numbers c_1 and c_2 in a way to account for the boundary condition $G_\pm(0) = 0$, and then calculate the S-matrix element from the asymptotic properties of this function. This calculation was carried out in appendix B, we merely give the final result:

$$S_{0,v,\pm}(k) = \pm 2^{2\mathrm{i}k}\frac{\Gamma(1/2+\mathrm{i}k)\Gamma((1/2+v+\mathrm{i}k)/2)\Gamma((3/2-v+\mathrm{i}k)/2)}{\Gamma(1/2-\mathrm{i}k)\Gamma((1/2+v-\mathrm{i}k)/2)\Gamma((3/2-v-\mathrm{i}k)/2)}. \qquad (4.20)$$

As we have already remarked, the results for the SUSY partner potentials are the same up to a minus sign. Note, however, that k in this formula is *complex*.

5. A realization of $SO(3,2)$ using coordinates on the coset $SO(3,2)/SO(3,1)$

In this section we present the realization found in our paper in [1] for the group $SO(3,2)$. As a first step as in section 3 we have to choose a finite-dimensional non-unitary matrix representation $\mathcal{D}$ of the subgroup $SO(3,1)$. Having chosen $\mathcal{D}$, we are given the *six* matrices S_1, S_2, S_3 and T_1, T_2, T_3 satisfying the $so(3,1)$ commutation relations

$$[S_i, S_j] = \mathrm{i}\varepsilon_{ijk}S_k \qquad [S_i, T_j] = \mathrm{i}\varepsilon_{ijk}T_k \qquad [T_i, T_j] = -\mathrm{i}\varepsilon_{ijk}S_k. \qquad (5.1)$$

By assumption the generators **S** are Hermitian and the **T** are anti-Hermitian, hence we have a non-unitary representation.

5.1. *Trivial inducing representation*

We represent the *four*-dimensional coset space $SO(3,2)/SO(3,1)$ as

$$\{X_a,,\ a = 1,\ldots,5| - X_1^2 - X_2^2 - X_3^2 + X_4^2 + X_5^2 = 1\}.$$

The $SO(3,2)$ invariant line element is

$$\mathrm{d}s^2 = -\mathrm{d}X_1^2 - \mathrm{d}X_2^2 - \mathrm{d}X_3^2 + \mathrm{d}X_4^2 + \mathrm{d}X_5^2 \qquad -\mathbf{X}^2 + X_4^2 + X_5^2 = 1.$$

We can obtain the usual $SO(3,2)$ generators corresponding to infinitesimal transformations preserving this line element in the usual way. The result is

$$\varepsilon_{jkm}L_m = X_jP_k - X_kP_j \qquad V = X_4P_5 - X_5P_4 \qquad (5.2a)$$

$$B_j = X_jP_4 + X_4P_j \qquad D_j = X_jP_5 + X_5P_j \qquad (5.2b)$$

where $P_a = -\mathrm{i}\partial_a,\ a = 1, \dots, 5.$ The commutation relations are

$$[L_m, L_n] = \mathrm{i}\varepsilon_{mnk} L_k \qquad [L_m, B_n] = \mathrm{i}\varepsilon_{mnk} B_k \qquad [L_m, D_n] = \mathrm{i}\varepsilon_{mnk} D_k \tag{5.3a}$$

$$[V, B_n] = \mathrm{i}D_n \qquad [V, D_n] = -\mathrm{i}B_n \tag{5.3b}$$

$$[B_m, B_n] = [D_m, D_n] = -\mathrm{i}\varepsilon_{mnk} L_k \qquad [B_m, D_n] = -\mathrm{i}\delta_{mn}. \tag{5.3c}$$

On our four-dimensional coset we introduce the usual stereographically projected coordinates (see also equation (3.4))

$$\mathbf{X} = \frac{2}{1 - R^2}\mathbf{R} \qquad X_4 = \frac{1 + R^2}{1 - R^2} v_1 \qquad X_5 = \frac{1 + R^2}{1 - R^2} v_2 \tag{5.4}$$

to obtain the line element identical in form to (3.5). Now $\mu, \nu = 1, \dots, 4$, $(y_1, y_2, y_3, y_4) \equiv (R_1, R_2, R_3, \chi)$, and $v_1 = \cos\chi$, $v_2 = \sin\chi$ as usual. The $SO(3, 2)$ generators in these coordinates are

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} \qquad V = v_3 \tag{5.5a}$$

$$\mathbf{B} = \mathbf{K}v_1 - \frac{2}{1 + R^2}\mathbf{R}v_2 v_3 \qquad \mathbf{D} = \mathbf{K}v_2 + \frac{2}{1 + R^2}\mathbf{R}v_1 v_3 \tag{5.5b}$$

where $\mathbf{K}$ has the same form as in (3.7); however, now just like $\mathbf{L}$ it has three components. Moreover, as in section 3, the six operators $\mathbf{L}$ and $\mathbf{K}$ generate an $SO(3, 1)$ algebra. Notice also that $\mathbf{L}$ and $\mathbf{K}$ are related to the angular momentum operator and the Runge–Lenz vector after a canonical and a similarity transformation [7]. Again as in section 3 we can say that our $SO(3, 2)$ algebra is built up by using the generators $(\mathbf{L}, \mathbf{K})$ of an $SO(3, 1)$ algebra.

One can calculate the quadratic Casimir, and find

$$C(SO(3, 2)) = \mathbf{L}^2 + V^2 - \mathbf{B}^2 - \mathbf{D}^2 = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu). \tag{5.6}$$

Moreover, by employing the similarity transformation (2.7) $T(R) = \sqrt{\frac{1-R^2}{1+R^2}}$ one can relate our $SO(3, 2)$ Casimir to the $SO(3, 1)$ Casimir $C(SO(3, 1)) = \mathbf{L}^2 - \mathbf{K}^2$ as follows:

$$T^{-1}C(SO(3, 2))T = C(SO(3, 1)) + \left(v_3^2 - \frac{1}{4}\right)\left(\frac{1 - R^2}{1 + R^2}\right)^2 - \frac{5}{4}. \tag{5.7}$$

By choosing scattering states for which the eigenvalue of the operator v_3 (the potential strength parameter) is $\frac{1}{2}$ the $SO(3, 2)$ Casimir describes all the scattering phenomena obtained for the $SO(3, 1)$ Casimir (for example, Coulomb scattering after a canonical transformation see [7, 18]). A great advantage of this geometrically motivated construction is that it substantially simplifies the analysis as given in [18].

Acting with this Casimir operator on the states $|\omega, l, m, v\rangle$ where $V|\omega, m, v\rangle = v|\omega, m, v\rangle$, $L^2|\omega, l, m, v\rangle = l(l + 1)|\omega, l, m, v\rangle$, and $L_3|\omega, l, m, v\rangle = m|\omega, l, m, v\rangle$, and $C|\omega, l, m, v\rangle = \omega(\omega + 3)|\omega, l, m, v\rangle$, and choosing the principal series of irreducible representations with $\omega = -3/2 + \mathrm{i}k$, we get a Schrödinger equation for a *three-dimensional* scattering problem with the potential

$$-\frac{l(l + 1)}{r^2} + \frac{l(l + 1)}{\sinh^2 \frac{r}{2}} - \frac{v^2 - \frac{1}{4}}{\cosh^2 \frac{r}{2}}$$

in agreement with [18].

5.2. Non-trivial inducing representations

Now we have to modify our (5.5) realization by adding matrix-valued modifications to the generators. For this purpose we use the observation (motivated by results of the previous sections) that an $so(3,1)$ algebra spanned by $(\mathbf{L},\mathbf{K})$ sits inside the realization (5.5). Moreover, we know from [7] how to modify $\mathbf{L}$ and $\mathbf{K}$:

$$\mathbf{J}=\mathbf{L}+\mathbf{S} \qquad \mathbf{M}=\mathbf{K}+\mathbf{S}\times\mathbf{R} \tag{5.8}$$

with the Hermitian $\mathbf{S}$ satisfying the first commutator of (5.1). One can check that these operators satisfy

$$[J_i,J_j]=\mathrm{i}\varepsilon_{ijk}J_k \qquad [J_i,M_j]=\mathrm{i}\varepsilon_{ijk}M_k \qquad [M_i,M_j]=-\mathrm{i}\varepsilon_{ijk}J_k \tag{5.9}$$

i.e. the commutation relations of the $so(3,1)$ algebra.

Our task is to find a similar realization for the Lie algebra of $SO(3,2)$. Looking at the (5.5) $SO(3,2)$ realization one can see that a straightforward way to modify it is to replace the $SO(3,1)$ generators $(\mathbf{L},\mathbf{K})$ by the modified ones $(\mathbf{J},\mathbf{M})$, and to add possible further modifications to them. The commutation relations that are to be satisfied are

$$[J_m,J_n]=\mathrm{i}\varepsilon_{mnk}J_k \qquad [J_m,\mathcal{B}_n]=\mathrm{i}\varepsilon_{mnk}\mathcal{B}_k \qquad [J_m,\mathcal{D}_n]=\mathrm{i}\varepsilon_{mnk}\mathcal{D}_k \tag{5.10a}$$

$$[V,\mathcal{B}_n]=\mathrm{i}\mathcal{D}_n \qquad [V,\mathcal{D}_n]=-\mathrm{i}\mathcal{B}_n \tag{5.10b}$$

$$[\mathcal{B}_m,\mathcal{B}_n]=[\mathcal{D}_m,\mathcal{D}_n]=-\mathrm{i}\varepsilon_{mnk}J_k \qquad [\mathcal{B}_m,\mathcal{D}_n]=-\mathrm{i}\delta_{mn}. \tag{5.10c}$$

Motivated by the results of section 3, one can show that the new generators are

$$\mathbf{J}=\mathbf{L}+\mathbf{S} \qquad V=v_3 \tag{5.11a}$$

$$\mathcal{B}=v_1\mathbf{M}-v_2f(R)\mathbf{R}(v_3-\mathbf{TR})-\mathbf{T}v_2 \tag{5.11b}$$

$$\mathcal{D}=v_2\mathbf{M}+v_1f(R)\mathbf{R}(v_3-\mathbf{TR})+\mathbf{T}v_1 \tag{5.11c}$$

where $f(R)$ is defined as in (3.13). Notice again that V was not modified.

The simplest choice for $\mathcal{D}$ used in [1] is

$$S_j\equiv\begin{pmatrix}\frac{1}{2}\sigma_j & 0\\ 0 & \frac{1}{2}\sigma_j\end{pmatrix} \qquad T_j\equiv\begin{pmatrix}0 & \frac{1}{2}\mathrm{i}\sigma_j\\ \frac{1}{2}\mathrm{i}\sigma_j & 0\end{pmatrix}. \tag{5.12}$$

There we calculated the Casimir operators for the realization (5.11), and the Schrödinger equation for the corresponding scattering problem was also derived. Since the details can be found in that paper, and the calculation follows the same steps as in section 3 for the $SO(2,2)$ case, we merely give the result for the interaction term:

$$V_{\alpha\beta}\equiv \begin{pmatrix} \dfrac{-l(l+1)}{r^2}+\dfrac{l(l+1)}{\sinh^2 r}-\dfrac{\kappa}{2\cosh^2\frac{r}{2}}-\dfrac{v^2}{\cosh^2 r} & -\mathrm{i}v\dfrac{\sinh r}{\cosh^2 r}\\ -\mathrm{i}v\dfrac{\sinh r}{\cosh^2 r} & \dfrac{-l(l+1)}{r^2}+\dfrac{l(l+1)}{\sinh^2 r}+\dfrac{\kappa}{2\cosh^2\frac{r}{2}}-\dfrac{v^2}{\cosh^2 r}\end{pmatrix} \tag{5.13}$$

where $\kappa=j+\frac{1}{2}$. It is easy to show that (5.13) can be rewritten in the form

$$V^{\kappa,v}(r)=-\frac{l(l+1)}{r^2}+\frac{\kappa^2+\kappa\sigma_3\cosh r}{\sinh^2 r}-\frac{v^2+\mathrm{i}v\sigma_1\sinh r}{\cosh^2 r} \tag{5.14}$$

which is of the form (3.28). This non-Hermitian interaction term yields the same type of radial Schrödinger equation. The discussion of the special solutions using section 4 is trivial.

6. Conclusions and comments

In this paper we have constructed matrix-valued realizations for the non-compact groups $SO(2,1)$, $SO(2,2)$ and $SO(3,2)$ frequently used in algebraic scattering theory (AST). Finding new realizations for the group $SO(3,2)$ is especially important, as this is a candidate for describing heavy ion reactions. Such reactions are described by non-local, spin-dependent, optical (complex) potentials. Previously it was demonstrated that non-local, spin-dependent potentials can indeed be obtained using AST. In our study we have shown that optical potentials are also amenable to an algebraic description.

Our basic tool was the theory of induced representations. According to this the geometry of the scattering problem is expressed in terms of coordinates on a coset space G/H. The vector-valued complex wavefunctions $\psi : G/H \to \mathbb{C}^n$ (more precisely wave sections of a vector bundle over G/H) carry an n-dimensional matrix representation $\mathcal{D}$ of H. The modified matrix-valued symmetry generators of G acting on these wavefunctions are the generators of the induced representation for G induced by $\mathcal{D}$. When H is a non-compact group, $\mathcal{D}$ can be chosen to be a non-unitary representation. Some of the generators in this case are a non-Hermitian matrix. Defining some suitable combinations of the Casimir operators of G to be the Hamiltonian H of some scattering problem will yield a non-Hermitian interaction term, i.e. optical potentials.

In this paper we illustrated this method by choosing G to be $SO(2,1)$, $SO(2,2)$ and $SO(3,2)$ with H being $SO(1,1)$, $SO(2,1)$ and $SO(3,1)$, respectively. We constructed the generators and calculated the Casimir operators. By extracting the interaction term we demonstrated that these are indeed complex potentials. These turned out to be of 2×2 matrices containing potentials of Pöschl–Teller and Gendenshtein type. For special cases we calculated the corresponding S-matrices.

It is quite natural to generalize this construction for the groups $SO(n,2)$. However, our $SO(3,2)$ example clearly exhibits the basic structure of the interaction matrix. (Compare equations (3.28) and (5.14).) Based on this observation we can say that provided we choose $\mathcal{D}$ to be the finite-dimensional spinor representation of $SO(n,1)$ (compare equations (3.17) and (5.12)), we obtain the same structure for the interaction term. Of course we can try finite-dimensional matrix representations for $SO(n,1)$ other than those based on Dirac gamma matrices (Clifford algebras). Such realizations would describe scattering processes with higher spin. These potentials might exhibit interesting new features (e.g., the appearance of a non-central interaction); however, according to our parametrization of our coset, the radial dependence will still be in terms of Pöschl–Teller and Gendenshtein-like potentials.

The question then arises: how does one obtain potentials of other types? One possibility is to choose other parametrizations for our cosets. More importantly, there is another method for obtaining new potentials from the expression of the Casimir operators. This is the method of canonical transformations [7, 18]. One can quickly illustrate the usefulness of this method by choosing $G = SO(2,2)$ and looking at the expression (3.18*b*) for its Casimir operator $\mathcal{C}'$. By a suitable similarity transformation depending only on R one can transform this to the form

$$\tilde{C}' = \frac{\mathrm{i}}{4}(\boldsymbol{\sigma} P)(1 - R^2) + \frac{1}{2}\sigma_3 v_3 \frac{1 - R^2}{1 + R^2}. \tag{6.1}$$

Now we employ the canonical transformation

$$\mathbf{R} \to -\frac{1}{\sqrt{2E}}\mathbf{P} \qquad \mathbf{P} \to \sqrt{2E}\mathbf{R} \tag{6.2}$$

leaving the commutation relations intact, hence arriving at the expression

$$2\tilde{C}'_{KAN} = \left(\frac{\mathrm{i}}{\sqrt{2E}} R(\boldsymbol{\sigma}\mathbf{n}) + \sigma_3 v_3 (E + P^2/2)^{-1}\right)(E - P^2/2) \tag{6.3}$$

where E is the eigenvalue (the scattering energy, possibly complex) of an as yet unknown Hamiltonian that commutes with the generators of $SO(2,2)$ obtained from (3.13) by using (6.2). From (6.3) we identify the term $E - P^2/2$ with the potential U, and after rearranging terms we get

$$-2\mathrm{i}\sqrt{2E}(\boldsymbol{\sigma}\mathbf{n})\tilde{C}'_{KAN} = \left(R + \sqrt{2E}(\boldsymbol{\sigma}\mathbf{n}^{\star})v(E + P^2/2)^{-1}\right)U \tag{6.4}$$

where $\mathbf{n}^{\star}$ is the dual of the two-component vector $\mathbf{n}$ (see equation (3.6)), and we used the eigenvalue of v_3 by evaluating (6.4) on the corresponding eigensubspace. By calculating the inverse of the operator standing in front of U, one can express the interaction term formally. In order to gain some insight we put $v = 0$ in (6.4). Then we have

$$U = -2\mathrm{i}\frac{\sqrt{2E}}{R}(\boldsymbol{\sigma}\mathbf{n})\tilde{C}'_{KAN}. \tag{6.5}$$

Let us recall equation (3.21*b*). We suppose that the Hamiltonian is some function of $\tilde{C}'$, then according to [7, 18] we rewrite (3.21*b*) as

$$\tilde{C}'|\pm\tfrac{1}{2}, f(k)\rangle = \pm\tfrac{1}{2}\mathrm{i}f(k)|\pm\tfrac{1}{2}, f(k)\rangle. \tag{6.6}$$

Choosing $f(k) = Z_1Z_2e^2/k = Z_1Z_2e^2/\sqrt{2E}$ (the Sommerfeld parameter), and acting with (6.5) on the scattering states defined by (6.6) with a similar reasoning then in [7] (notice that $\tilde{C}'$ and $\boldsymbol{\sigma}\mathbf{n}$ for $v = 0$ are both parity odd operators, hence U should be parity even) we get

$$U(R) = \pm\frac{Z_1Z_2e^2}{R} \tag{6.7}$$

i.e. the Coulomb potential. This result should not come as a surprise, since by setting $v = 0$ the $SO(2,2)$ symmetry of the problem was reduced to the group $SO(2,1)$, which is known to be the symmetry group of the two-dimensional Coulomb problem.

For $v \neq 0$ one can try to solve the equation

$$\pm\frac{\alpha}{R} = (I - \lambda K)U \qquad K \equiv (\boldsymbol{\sigma}\mathbf{n}^{\star})\frac{1}{R}(k^2 + P^2)^{-1} \tag{6.8}$$

where $\lambda \equiv -2kv,\ k = \sqrt{2E} \in \mathbb{C}$, and $\alpha \equiv Z_1Z_2e^2$. Here equation (6.8) is sensible only when acting on the eigenstates defined by (6.6). The formal solution of this equation is given by expanding $(I - \lambda K)^{-1}$ in a power series in λ, i.e

$$U = \pm(I - \lambda K)^{-1}\frac{\alpha}{R} = \pm\alpha\sum_{n=0}^{\infty}\lambda^n K^n\frac{1}{R}. \tag{6.9}$$

The question is, of course: under what circumstances is such a formal expansion legitimate? Such issues will be addressed in a subsequent publication. Here we would merely like to point out in closing that for small v equation (6.9) results in the expression (($\boldsymbol{\sigma}\mathbf{n}^{\star}$) acts on the states like (3.25) as the matrix $-\sigma_2$)

$$\pm U = \frac{\alpha}{R}I + 2v\frac{\alpha}{R}\frac{k}{k^2 + P^2}\frac{1}{R}\sigma_2 + \cdots \tag{6.10}$$

which is an energy-dependent two channel, non-local optical potential of modified Coulomb type. Such algebraic potentials are the generalizations of the ones that can be found in [7, 18]. The non-local nature of the operator can be made manifest by representing K as an integral operator with kernel defined by the Green function of the Helmholtz operator $(P^2 + k^2)$, $k \in \mathbb{C}$ in *two* dimensions.

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Appendix A. Properties of the modified $SO(2, 1)$ generators

In this appendix we try to give some insight into the meaning of the modified generators of (2.9). For this purpose we consider the vector **W** with components $(W_1, W_2, W_3) = q\,\frac{1-R^2}{1+R^2}(v_1, v_2, 0)$ modifying the original set of generators. Moreover, we write the generators $L_k,\ k = 1, 2, 3$ in the form $L_k = g_k^{\mu} P_{\mu},\ \mu = 1, 2$, with $P_1 = -\mathrm{i}\partial_R$ and $P_2 = -\mathrm{i}\partial_{\chi}$, where the functions $g_k^{\mu}(R, \chi)$ are defined appropriately. In this notation the modified $SO(2, 1)$ generators can be expressed as $J_k = L_k + W_k,\ k = 1, 2, 3$.

From (2.10) it is obvious that up to the constant q^2 the modification in the Casimir is effected by the presence of the vector field A_{μ}. Let us examine how A_{μ} behaves under an infinitesimal $SO(2, 1)$ rotation generated by the operators **L**. The change in A_{μ} is $\mathcal{L}_g A_{\mu} \equiv g_k^{\nu}\partial_{\nu} A_{\mu} + (\partial_{\mu} g_k^{\nu}) A_{\nu}$ which is called the *Lie derivative* of A_{μ}. Using the explicit form of A from (2.11) one can prove that

$$\mathcal{L}_{g_k} A_{\mu} = \partial_{\mu} W_k \qquad k = 1, 2, 3. \tag{A.1}$$

The meaning of this equation is as follows. The vector field A_{μ} is *not* invariant under the kth infinitesimal $SO(2, 1)$ transformations, but it is invariant *up to a compensating gauge transformation* $\partial_{\mu} W_k$ (the gradient of a scalar function for each k). Hence we have clarified the meaning of the W_k modifications.

According to the general theory of induced representations [3] we have a group G, a subgroup H and a finite-dimensional irreducible representation $\mathcal{D}$ for the subgroup. One then considers the coset space G/H; in our case it is the one-sheet hyperboloid $SO(2, 1)/SO(1, 1)$. Then one constructs the vector bundle with the base space being G/H and the fibre being the carrier space for the representation $\mathcal{D}$ which is a finite-dimensional vector space. One can imagine this construction as attaching a copy of the representation space to each point in G/H. We have to emphasize that this construction is local, in the sense that our vector bundle is *not* the Cartesian product of G/H and our finite-dimensional vector space. The (local) sections of this vector bundle are vector-valued wavefunctions carrying the representation $\mathcal{D}$. In our case these are one-component wavefunctions, since $SO(1, 1)$ is Abelian. The induced representation is defined by defining an action of G on such sections ('wavefunctions'), see [3] for details. On our bundle we can naturally define a rule for comparing wavefunctions at different points of G/H. This rule defines a *parallel transport* for the wavefunctions. For our case it is precisely the vector potential A_{μ} which defines this rule. Our modified symmetry generators turn out to be the infinitesimal operators of the induced representation acting on wavefunctions. The difference between these operators and the usual ones (**L**) are compensating gauge transformations for the gauge field defining the parallel transport (see equation (A.1)).

Appendix B. Transformations of the wavefunction for the Gendenshtein potential

The wavefunction satisfying the Schrödinger equation with the Gendenshtein potential

$$V(r) = \frac{q^2 - v^2 - 2qv\sinh r + \frac{1}{4}}{\cosh r^2}. \tag{B.1}$$

is the linear combination [11]

$$\psi_{k,v,q}(r) = E_{v,q}(r)\big(c_1 F(1/2 + v + \mathrm{i}k, 1/2 + v - \mathrm{i}k, 1 + v - \mathrm{i}q, z(r)) + c_2 z(r)^{-v+\mathrm{i}q} F(1/2 + \mathrm{i}q + \mathrm{i}k, 1/2 + \mathrm{i}q - \mathrm{i}k, 1 - v + \mathrm{i}q, z(r))\big) \tag{B.2}$$

where

$$z(r) = \tfrac{1}{2}(1 + \mathrm{i}\sinh r) \qquad E_{v,q} = (\cosh r)^{\frac{1}{2}+v}\exp\left(q\tan^{-1}\sinh r\right). \tag{B.3}$$

We are particularly interested in the cases with $q = \pm\frac{\mathrm{i}}{2}$. Since the potentials of these cases are SUSY partners of each other, we set $q = -\frac{\mathrm{i}}{2}$. The S-matrix of the other case is obtained simply by multiplying by -1. In first case we have to consider the linear combination of two hypergeometric functions in the form

$$\Phi = c_1\mathcal{F}_1 + c_2 z^{-v-1/2}\mathcal{F}_2 \equiv c_1 F(1/2 + v + \mathrm{i}k, 1/2 + v - \mathrm{i}k, 3/2 + v, z) + c_2 z^{-v-1/2} F(+\mathrm{i}k, -\mathrm{i}k, 1/2 - v, z) \tag{B.4}$$

which are just our functions also derived in (4.18). We would like to choose the complex numbers c_1 and c_2 in such a way as to account for the boundary condition $\psi(0) = 0$. For this purpose we apply the transformation formula [17]

$$(a - b)F(a, b, c, z) = aF(a + 1, b, c, z) - bF(a, b + 1, c, z) \tag{B.5}$$

to arrive at the following form of our functions:

$$\mathcal{F}_1 = -\frac{\mathrm{i}}{2k}\big[(1/2 + v + \mathrm{i}k)F(3/2 + v + \mathrm{i}k, 1/2 + v - \mathrm{i}k, 3/2 + v, z) - (1/2 + v - \mathrm{i}k)F(1/2 + v + \mathrm{i}k, 3/2 + v - \mathrm{i}k, 3/2 + v, z)\big] \tag{B.6a}$$

$$\mathcal{F}_2 = \tfrac{1}{2}\big[F(1 + \mathrm{i}k, -\mathrm{i}k, 1/2 - v, z) + F(\mathrm{i}k, 1 - \mathrm{i}k, 1/2 - v, z)\big]. \tag{B.6b}$$

As a next step, we get rid of the variable $z(r) = \frac{1}{2}(1 + \mathrm{i}\sinh r)$ by transforming it into $w(r) \equiv 4z(r)(1 - z(r)) = \cosh^2 r$ using the expressions [17]

$$F(a, b, (a + b + 1)/2, z) = F(a/2, b/2, (a + b + 1)/2, 4z(1 - z)) \tag{B.7a}$$

$$F(a, 1 - a, c, z) = (1 - z)^{c-1}F((c - a)/2, (c + a - 1)/2, c, 4z(1 - z)) \tag{B.7b}$$

with the result

$$\mathcal{F}_1 = -\frac{\mathrm{i}}{2k}\big[(1/2 + v + \mathrm{i}k)F((3/2 + v + \mathrm{i}k)/2, (1/2 + v - \mathrm{i}k)/2, 3/2 + v, w) - (1/2 + v - \mathrm{i}k)F((1/2 + v + \mathrm{i}k)/2, (3/2 + v - \mathrm{i}k)/2, 3/2 + v, w)\big] \tag{B.8a}$$

$$z^{-v-1/2}\mathcal{F}_2 = \tfrac{1}{2}(\tfrac{1}{2}w)^{-v-1/2}\big[F((-1/2 - v - \mathrm{i}k)/2, (1/2 - v + \mathrm{i}k)/2, 1/2 - v, w) + F((1/2 - v - \mathrm{i}k)/2, (-1/2 - v + \mathrm{i}k)/2, 1/2 - v, w)\big]. \tag{B.8b}$$

We can transform these expressions further using the formula [17]

$$F(a, b, c, w) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}F(a, b, a + b - c + 1, 1 - w) + (1 - w)^{c-a-b}\frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}F(c - a, c - b, c - a - b + 1, 1 - w) \tag{B.9}$$

and then grouping similar terms employing the identity [17]

$$\Gamma(a, b, c, 1-w) = w^{c-a-b} F(c-a, c-b, c, 1-w) \tag{B.10}$$

hence arriving at the following formula for the function $\Phi \equiv c_1 \mathcal{F}_1 + c_2 z^{v-1/2} \mathcal{F}_2$:

$$\Phi(r) = \Phi_1(r) + \mathrm{i} \sinh r (\cosh r)^{-2v-1} \Phi_2(r) \tag{B.11}$$

with

$$\begin{aligned}
\Phi_1 = {} & \left(-\frac{\mathrm{i}}{k} c_1 \Gamma_1 + 2^{2v} c_2 \Gamma_3 \right) F((3/2 + v + \mathrm{i}k)/2, (1/2 + v - \mathrm{i}k)/2, 1/2, -\sinh^2 r) \\
& + \left(\frac{\mathrm{i}}{k} c_1 \Gamma_2 + 2^{2v} c_2 \Gamma_4 \right) \\
& \times F((1/2 + v + \mathrm{i}k)/2, (3/2 + v - \mathrm{i}k)/2, 1/2, -\sinh^2 r)
\end{aligned} \tag{B.12a}$$

$$\begin{aligned}
\Phi_2 = {} & \left(-\frac{\mathrm{i}}{k} c_1 \Omega_1 + 2^{2v} c_2 \Omega_3 \right) F((3/2 - v + \mathrm{i}k)/2, (1/2 - v - \mathrm{i}k)/2, 3/2, -\sinh^2 r) \\
& + \left(\frac{\mathrm{i}}{k} c_1 \Omega_2 + 2^{2v} c_2 \Omega_4 \right) F((1/2 - v + \mathrm{i}k)/2, (3/2 - v - \mathrm{i}k)/2, 3/2, -\sinh^2 r) \\
\equiv {} & \alpha_1 F_1 + \alpha_2 F_2
\end{aligned} \tag{B.12b}$$

where

$$\Gamma_1(k, v) = \Gamma_2(-k, v) = \frac{\Gamma(3/2 + v)\Gamma(1/2)}{\Gamma((3/2 + v - \mathrm{i}k)/2)\Gamma((1/2 + v + \mathrm{i}k)/2)} \tag{B.13a}$$

$$\Gamma_3(k, v) = \Gamma_4(-k, v) = 2(-1/2 - v + \mathrm{i}k)^{-1} \Gamma_1(k, -1 - v) \tag{B.13b}$$

$$\Omega_3(k, v) = \Omega_4(-k, v) = \frac{\Gamma(1/2 - v)\Gamma(-1/2)}{\Gamma((-1/2 - v - \mathrm{i}k)/2)\Gamma((1/2 - v + \mathrm{i}k)/2)} \tag{B.14a}$$

$$\Omega_1(k, v) = \Omega_2(-k, v) = \tfrac{1}{2}(1/2 + v + \mathrm{i}k)\Omega_3(k, -1 - v). \tag{B.14b}$$

Since $E_{v,\frac{1}{2}\mathrm{i}}(0) \neq 0$, our wavefunction vanishes when $\Phi(r)$ does. This means that $\Phi_1(r)$ in (B.11) must vanish, yielding the constraint

$$-\frac{\mathrm{i}}{k} c_1 (\Gamma_1 - \Gamma_2) + 2^{2v} c_2 (\Gamma_3 + \Gamma_4) = 0 \tag{B.15}$$

where we made use of the fact that $F(a, b, c, 0) = 1$. Then we are left with Φ_2, and the factors multiplying the hypergeometric functions in (B.12b) are ($\Phi_2 = \alpha_1 F_1 + \alpha_2 F_2$)

$$\alpha_1(k, v) = \frac{\Omega_1}{\Gamma_1 - \Gamma_2} - \frac{\Omega_3}{\Gamma_3 + \Gamma_4} \qquad \alpha_2(k, v) = \frac{\Omega_2}{\Gamma_2 - \Gamma_1} - \frac{\Omega_4}{\Gamma_3 + \Gamma_4}. \tag{B.16}$$

It is easy to check that $\alpha_2(k, v) = \alpha_1(-k, v)$, hence it is enough to calculate α_1. A straightforward calculation using equations (B.13), (B.14) shows that

$$\alpha_1(k, v) = (1/2 + v + \mathrm{i}k)[(1 - Q(k, v))^{-1} - (1 + Q(-k, -v))^{-1}] \tag{B.17}$$

where

$$Q(k, v) = \frac{\Gamma((3/2 + v + \mathrm{i}k)/2)\Gamma((1/2 + v - \mathrm{i}k)/2)}{\Gamma((3/2 + v - \mathrm{i}k)/2)\Gamma((1/2 + v + \mathrm{i}k)/2)}. \tag{B.18}$$

We will also need the ratio α_1/α_2, which turns out to be

$$\frac{\alpha_1(k, v)}{\alpha_2(k, v)} = (-1)\frac{1/2 + v + \mathrm{i}k}{1/2 + v - \mathrm{i}k}. \tag{B.19}$$

The next step is to transform the functions F_1 and F_2 in (B.12*b*), in order to extract their asymptotic behaviour. For this purpose we refer to the formulae [17]

$$F(a,b,c,1-w) = w^{-a}F(a,c-b,c,(w-1)/w) = w^{-b}F(b,c-a,c,(w-1)/w). \tag{B.20}$$

Then for the functions $F_{1,2}$ we get the expressions

$$F_1(k,v,r) = (\cosh r)^{-(1/2-v-\mathrm{i}k)}F((1/2-v-\mathrm{i}k)/2,(3/2+v-\mathrm{i}k)/2,3/2,\tanh r) \tag{B.21a}$$

$$F_2(k,v,r) = (\cosh r)^{-(1/2-v+\mathrm{i}k)}F((1/2-v+\mathrm{i}k)/2,(3/2+v+\mathrm{i}k)/2,3/2,\tanh r). \tag{B.21b}$$

Finally, using these, we obtain the following form for our wavefunction:

$$\psi_{k,v,\frac{1}{2}\mathrm{i}}(r) = E_{v,\frac{1}{2}\mathrm{i}}(r)[\alpha_1(k,v)F_1(k,v,r)+\alpha_2(k,v)F_2(k,v,r)] \tag{B.22}$$

where the corresponding quantities are defined by (B.3), (B.17)–(B.19) and (B.21). As $r\to\infty$, $\tanh r\to 1$: hence $F(a,b,c,\tanh r)\to\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$ [17]. Collecting everything, we get

$$\psi_{k,v,\frac{1}{2}\mathrm{i}} \to \alpha_1(k,v)A(k,v)2^{-\mathrm{i}k}\mathrm{e}^{\mathrm{i}kr}+\alpha_2(k,v)A(-k,v)2^{\mathrm{i}k}\mathrm{e}^{-\mathrm{i}kr} \tag{B.23}$$

where

$$A(k,v) = \frac{2\Gamma(1/2+\mathrm{i}k)\Gamma(3/2)}{\Gamma((1/2+v+\mathrm{i}k)/2)\Gamma((3/2-v+\mathrm{i}k)/2)}(1/2+v+\mathrm{i}k)^{-1}. \tag{B.24}$$

Now we can read off the reflection coefficent $\mathcal{R}$ which is related to the S-matrix as $S_v(k) = -\mathcal{R}_v(k)$. The final result is

$$S_v(k) = 2^{2\mathrm{i}k}\frac{\Gamma(1/2+\mathrm{i}k)\Gamma((1/2+v+\mathrm{i}k)/2)\Gamma((3/2-v+\mathrm{i}k)/2)}{\Gamma(1/2-\mathrm{i}k)\Gamma((1/2+v-\mathrm{i}k)/2)\Gamma((3/2-v-\mathrm{i}k)/2)}. \tag{B.25}$$

Moreover, as we have already remarked, the result for the SUSY partner potential is the same up to a minus sign.

References

[1] Apagyi B, Endrédi G and Lévay P (ed) 1997 *Inverse and Algebraic Quantum Scattering Theory: Proc. Conf. (Lake Balaton, Hungary, 1996)* (Berlin: Springer)
[2] Wu J, Iachello F and Alhassid Y 1987 *Ann. Phys., NY* **173** 68–87
[3] Lévay P 1994 *J. Phys. A: Math. Gen.* **27** 2857
[4] Lévay P and Apagyi B 1993 *Phys. Rev.* A **47** 823
[5] Lévay P and Apagyi B 1995 *J. Math. Phys.* **36** 6633
[6] Lévay P 1995 *J. Phys. A: Math. Gen.* **28** 5919
[7] Lévay P 1997 *J. Phys. A: Math. Gen.* **30** 7243
[8] Jones P B 1963 *The Optical Model in Nuclear and Particle Physics* (New York: Interscience)
[9] Barut A O and Raczka R 1977 *Theory of Group Representations and Applications* (Warsaw: Polish Scientific)
[10] Alhassid Y, Gürsey F and Iachello F 1986 *Ann. Phys., NY* **167** 181–200
[11] Khare A and Sukhatme U P 1988 *J. Phys. A: Math. Gen.* **21** L501–8
[12] Baye D, Lévai G and Sparenberg J M 1996 *Nucl. Phys.* A **599** 435–56
[13] Alhassid Y, Iachello F and Levine R D 1985 *Phys. Rev. Lett.* **54** 1746–9
[14] Wu J, Alhassid Y, and Gürsey F 1989 *Ann. Phys., NY* **196** 163–81
[15] Vilenkin N J and Klimyk A U 1991 *Representation of Lie Groups and Special Functions* vol 1 (Dordrecht: Kluwer)
[16] Vilenkin N J 1968 *Special Functions and the Theory of Group Representations* (Providence, RI: American Mathematical Society) ch VI.4.5
[17] Abramowitz M and Stegun I A 1964 *Handbook of Mathematical Functions* 5th edn (New York: Dover)
[18] Zielke A and Scheid W 1993 *J. Phys. A: Math. Gen.* **26** 2047